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Casimir buoyancy

Robert L. Jaffe and Antonello Scardicchio
Center for Theoretical Physics, Laboratory for Nuclear Sciences and Physics Department
Massachusetts Institute of Technology
Cambridge, MA 02139, U.S.A.
E-mail: jaffe@mit.edu, scardicc@mit.edu

Abstract: We study the Casimir force on a *single* surface immersed in an inhomogeneous medium. Specifically we study the vacuum fluctuations of a scalar field with a spatially varying squared mass, $m^2 + \lambda\Delta(x-a) + V(x)$, where V is a smooth potential and $\Delta(x)$ is a unit-area function sharply peaked around $x = 0$. $\Delta(x-a)$ represents a semi-penetrable thin plate placed at $x = a$. In the limits $\{\Delta(x-a) \to \delta(x-a),\ \lambda \to \infty\}$ the scalar field obeys a Dirichlet boundary condition, $\phi = 0$, at $x = a$. We formulate the problem in general and solve it in several approximations and specific cases. In all the cases we have studied we find that the Casimir force on the plate points in the direction opposite to the force on the quanta of ϕ: it pushes the plate toward higher potential, hence our use of the term *buoyancy*. We investigate Casimir buoyancy for weak, reflectionless, or smooth $V(x)$, and for several explicitly solvable examples. In the semiclassical approximation, which seems to be quite useful and accurate, the Casimir buoyancy is a *local* function of $V(a)$. We extend our analysis to the analogous problem in n-dimensions with $n-1$ translational symmetries, where Casimir divergences become more severe. We also extend the analysis to non-zero temperatures.

Keywords: Nonperturbative Effects, Boundary Quantum Field Theory.

Contents

1. Introduction

The interaction of a quantum field with a material medium sometimes can be idealized by placing a boundary condition on the field at the interface with the medium. Then the effects of the medium can be interpreted as modifications of the zero point energy of the field due to the boundary condition on the surface. The classic example is the force between two parallel, grounded conducting plates due to zero point fluctuations of the electromagnetic field first discovered by Casimir [1]. Hence problems of this nature are known in general as Casimir problems. If the surface $\mathcal{S}$ is embedded in an inhomogeneous medium, then the quantum fluctuations of ϕ sense the inhomogeneities and give rise to a force per unit area on the surface even in the absence of a second surface. We will refer to this force as *Casimir buoyancy*. As "buoyancy" implies, the force is opposite to the force that acts on the quanta of the fluctuating field, at least in the cases we have been able to study.

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In this paper we examine the problem of Casimir buoyancy. We formulate and study the problem for the simplest possible cases. We consider a scalar quantum field, ϕ, obeying the boundary conditions imposed by a sharply peaked background on an $n-1$ dimensional hyperplane embedded in n-dimensional euclidean space. We assume that the mass of the scalar field varies in the direction normal to the surface, i.e. we introduce an interaction $\frac{1}{2}V(x_\perp)\phi^2$ into the lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}V(x_\perp)\phi^2 - \frac{1}{2}\lambda\Delta(x_\perp - a)\phi^2 \,. \tag{1.1}$$

The function Δ, normalized to $\int dx\Delta(x) = 1$, is assumed to be sharply peaked at $x_\perp = a$. Usually a Dirac δ-function will do. We are particularly interested in the "Dirichlet limit", where the coupling λ goes to infinity and the field obeys the Dirichlet condition, $\phi = 0$, on the hyperplane. It is easy to imagine problems to which such a formulation applies, where a quantum field is subject to forces on two different scales, forces at a high energy scale that can be idealized as a boundary condition, and forces of order the mass of ϕ that can be regarded as a smoothly varying background.[1] We believe that similar considerations apply to a gauge vector (e.g. electromagnetic) field in an inhomogeneous medium and to fermion fields. The generic problem of a quantum field constrained on a surface and modulated by other forces in the bulk also arises in brane world scenarios, where similar effects should also be expected. We have not considered buoyancy in a curved space-time where the effect could arise from an inhomogeneous curvature [3].

As usual in Casimir physics, there is little intuition to guide us *a priori*. For example, there is no reason to expect the buoyancy force at a point x to depend only on V at the point x. In general we find that the buoyancy depends non-locally on V. However when the background field is smooth enough to admit a WKB approximation we find that the buoyancy reduces to a local function of the background field. Likewise we know of no argument to give the sign of the Casimir buoyancy. Should it be parallel to the force on the quanta of ϕ, or antiparallel? In all the examples we have been able to study we find that the force is opposite to the force on the quanta, hence a buoyancy.

Casimir problems can suffer from divergences of two different kinds.[2] The first are the familiar divergences that afflict any quantum field theory. The Casimir energy for a fluctuating field ϕ in a time-independent background[3] $\sigma(x)$ is the full one-loop effective energy, $E[\sigma]$, the sum over all one loop Feynman diagrams with arbitrary insertions of σ, as shown in the first line of figure 1 [4]. The low order Feynman diagrams diverge: the 1-point function diverges for $n \geq 1$, the two point function for $n \geq 3$, etc. As usual, these divergences are cancelled by counterterms that are polynomials in σ, $\mathcal{L}_{\text{CT}} = c_1\sigma + \frac{1}{2}c_2\sigma^2 + \cdots$ (some are shown in figure 1). The Casimir energy becomes dependent on the renormalized parameters of the σ-field dynamics. For example the need for the counterterm $\frac{1}{2}c_2\sigma^2$ in three dimensions generates a dependence on the renormalized σ-mass.

[1] In the *real* Casimir effect the idealization of the influence of the metal or dielectric on the electromagnetic field as a static background must be taken with some care. In particular some divergencies arising from this idealization could be absent if a dynamic description of the material is adopted [2].

[2] Both kinds of divergences are discussed in detail in ref. [5].

[3] For us $\sigma(x) = \lambda\delta(x-a) + V(x)$.

The second type of divergence is more interesting and more challenging. A realistic material medium cannot be idealized by a boundary condition at all energy scales. When the frequency of the fluctuating field is high compared to the natural scale of the interactions that characterize the material, its effects fade away. A background that constrains *all* modes of a fluctuating field is unphysical, and can introduce divergences into physical observables that cannot be removed by standard renormalization methods [5]. The origin of these divergences is quite clear from the Feynman diagrams. Each diagram involves integrals over the momenta carried by the external lines, $\sigma(x) = \int dp e^{-ipx}\hat{\sigma}(p)$. If the background has a discontinuity in some derivative, say the k-th, then at large p, $\hat{\sigma}(p) \sim p^{-k}$ and the integrals diverge for sufficiently large number of dimensions. There are no renormalization counter terms available to cancel these divergences. They signify that the quantity under consideration, even a directly measurable one like the Casimir force, is sensitive to the high energy cutoffs, Ω, above which the material no longer affects the field (say $\hat{\sigma}(p) = 0$ for $p > \Omega$).

In a given model there may be some quantities which admit a Casimir (i.e. boundary condition) description and others that do not. The boundary condition idealization shares with the effective field theory the notion of a separation of scales. The material structure is characterized by a high energy scale, the cutoff Ω. Modes with energies below this scale obey a boundary condition, modes with energies at or above Ω do not. If the boundary condition method is applicable, then physics at energies much lower than Ω can be described by the boundary condition without reference to Ω at all. For Casimir's original problem, parallel conducting plates, the plasma frequency ω_{plasma} sets the high energy scale $\Omega \sim \hbar\omega_{\text{plasma}}$, and the plate separation, d, or rather $\hbar c/d$, is the energy scale of physical interest. When $\omega_{\text{plasma}} \gg c/d$ the force between parallel plates is well described by the boundary condition calculation. It can be shown that the Casimir idealization works for Casimir forces between rigid bodies in vacuum and for local observables like the energy density outside the material (at distances greater than $\hbar c/\Omega$), two examples of immense practical importance. Other observables are not so fortunate. It was shown in ref. [5] that the Casimir *pressure* on a sphere of radius R, for example, depends on the cutoff, so it is not possible to study the pressure, even when $\hbar c/R \ll \Omega$, without characterizing the material in detail.

It is important to learn the circumstances under which there is an effective low energy description of buoyancy independent of the material cutoffs. We find that the answer to this question depends on the number of dimensions, n. In one dimension, where the surface is a point, there are no divergences of any kind and the Casimir buoyancy is independent of the cutoffs. Thus, for example, it goes to a finite limit as $\lambda \to \infty$. Higher dimensions can be studied relatively easily using the formalism developed in refs. [6]. For $n < 2$ the situation is the same as for $n = 1$. For $2 \le n < 3$, the Casimir buoyancy remains independent of the details of the surface, but the limit $\lambda \to \infty$ cannot be taken (it diverges like λ^{n-2}). For $n \ge 3$ there is no separation of scales. The buoyancy depends on the details of the structure of the material.

In the next section we describe the formulation of the Casimir buoyancy problem in one dimension. We recast it in terms of the Schrödinger equation Greens function with potential $V(x)$, and express the buoyancy in terms of the bound and scattering states

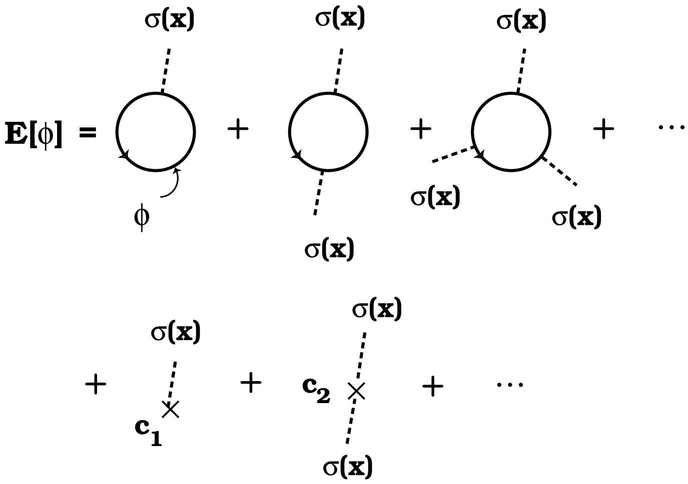

Figure 1: The Casimir energy of a fluctuating field, ϕ, coupled to a time independent background field, σ, via $\mathcal{L}_I = \frac{1}{2}\sigma(x)\phi^2$ is proportional to the sum of all one loop diagrams. The sum must include the contributions of counter terms, polynomials in σ, required to cancel the loop divergences. The structure of the counter terms depends on the number of dimensions, n. The second line in the figure shows the counter terms required in three dimensions where the 1- and 2- point functions are primitively divergent.

of V. We study both fixed λ and the limit $\lambda \to \infty$. In section 3 we describe some important approximations and special cases: For deep and smooth $V(x)$ we derive a WKB approximation; we study the case where $V(x) + m^2 = 0$ (a turning point) which cannot be analyzed by means of WKB, proving buoyancy in this case as well. We study the Casimir buoyancy force in a thermal state finding the buoyancy is qualitatively not affected by a non-zero temperature. For a reflectionless potential we show that only the bound states matter. We study the buoyancy outside the range of $V(x)$ and construct the first Born approximation as an example of non-local but simple result. In section 4 we go through some explicit, solvable examples. Then in section 5 we generalize from $n = 1$ to higher dimensions.

2. Formulation of the problem

The situation of interest is summarized by eq. (1.1). It might arise if a quantum field is coupled to one field characterized by a high mass scale that lives on the hyperplane $x_\perp = 0$, and to another field V characterized by a lower energy scale. As usual we take V to be time independent and externally determined; we ignore the back reaction of ϕ on V. The principal dynamical effects occur in the one non-trivial direction along which V varies, so we begin by studying the one-dimensional problem.

2.1 General considerations

In one dimension the lagrangian for ϕ reads

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - \frac{1}{2}\left(m^2 + \lambda\Delta(z-a) + V(z)\right)\phi^2\,. \tag{2.1}$$

We are interested in the vacuum energy of the ϕ-field as a functional of Δ and V,

$$\mathcal{E} = \frac{1}{2}\hbar \sum\!\!\!\!\!\!\!\int \omega = \frac{1}{2}\hbar \int_0^\infty dE\sqrt{E}\,\frac{dN}{dE}\,, \tag{2.2}$$

a sum over the discrete spectrum, if any, and integral over the continuum, where the $\{\omega^2\}$ are the eigenfrequencies of the Schrödinger hamiltonian,

$$\mathcal{H} = -\frac{d^2}{dx^2} + m^2 + \lambda\Delta(x-a) + V(x)\,, \tag{2.3}$$

which will, in general, range over both discrete and continues values.[4] So the problem can be recast in the form of a Schrödinger equation, $\mathcal{H}\phi = E\phi$, with $E = \omega^2 \equiv k^2 + m^2$. If we define $\mathcal{G}(x, x', E) = \langle x'|\frac{1}{\mathcal{H}-E}|x\rangle$ to be the equivalent Schrödinger Greens function, then we can write the density of states, dN/dE, as

$$\frac{dN}{dE} = \frac{1}{\pi}\,\mathrm{Im}\int d^{\,n}x\,\mathcal{G}(x, x, E + i\epsilon) \tag{2.4}$$

where n is the number of spatial dimensions. The $i\epsilon$ displaces the poles in $\mathcal{G}$ into the lower half E-plane, and insures that the propagator, $\mathcal{G}(x, x', t) = \int dE e^{-iEt}\mathcal{G}(x, x', E + i\epsilon)$, is causal. Substituting for the density of states into eq. (2.2), the Casimir energy can be written

$$\mathcal{E} = \frac{1}{2\pi}\,\mathrm{Im}\int_0^\Omega dE\sqrt{E}\int d^n x\mathcal{G}(x, x, E + i\epsilon)\,, \tag{2.5}$$

where we have assumed that the spectrum is positive definite[5] and we have introduced a cutoff Ω above which the boundary must be characterized in greater detail. We already discussed in the introduction the conditions (on n and $\sigma = \lambda\Delta + V$) under which we can take $\Omega \to \infty$ without affecting the low energy physics. We will go over some these arguments again in section 5. At the moment we assume all these conditions to be fulfilled so we can take the limit $\Omega \to \infty$ without placing further restrictions on $\sigma(x)$.

2.2 Force on a sharp surface

We are interested in the case where $\Delta(x-a)$ is a Dirac δ-function, $\Delta(x) = \delta(x)$, and $\lambda > 0$ since this repels the field ϕ from the "surface" $x = a$. If we restrict the analysis to 1 spatial dimension (the extension to $n > 1$ with translational symmetry in $n-1$ dimensions will be performed in section 5) it is possible to express the Greens function $\mathcal{G}$ in terms of the (simpler) Greens function, $\mathcal{G}_0(x', x, E) \equiv \langle x'|\frac{1}{\mathcal{H}_0-E}|x\rangle$ in the presence of $V(x)$ alone,

$$\mathcal{H}_0 = -\frac{d^2}{dx^2} + m^2 + V(x)\,. \tag{2.6}$$

First, we write the Lippmann-Schwinger equation[6] for $\mathcal{G}$ in terms of $\mathcal{G}_0$ and $\Delta(x)$,

$$\mathcal{G}(x, x', E) = \mathcal{G}_0(x, x', E) - \lambda\int dy\mathcal{G}_0(x, y, E)\Delta(y-a)\mathcal{G}(y, x', E)\,. \tag{2.7}$$

[4] We set $\hbar = c = 1$ until further notice.

[5] Negative energy single particle states in an external field correspond to vacuum instabilities that we do not consider here.

[6] Here we are considering a fully renormalized hamiltonian $\mathcal{H}$, a function of renormalized masses and couplings.

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When $\Delta(x-a) \to \delta(x-a)$ it is easy to solve for $\mathcal{G}$,[7]

$$\mathcal{G}(x,x',E) = \mathcal{G}_0(x,x',E) - \lambda \frac{\mathcal{G}_0(x,a,E)\mathcal{G}_0(a,x',E)}{1+\lambda\mathcal{G}_0(a,a,E)}\,. \tag{2.8}$$

For the density of states, we require the integral over x of $\mathcal{G}(x,x,E)$. Using the identity

$$\begin{aligned}\int dx \mathcal{G}_0(x,a,E)\mathcal{G}_0(a,x,E) &= \int dx \langle x|\frac{1}{\mathcal{H}_0 - E}|a\rangle\,\langle a|\frac{1}{\mathcal{H}_0 - E}|x\rangle \\ &= \langle a|\frac{1}{(\mathcal{H}_0 - E)^2}|a\rangle\,, \\ &= \frac{\partial}{\partial E}\mathcal{G}_0(a,a,E)\end{aligned} \tag{2.9}$$

we obtain

$$\begin{aligned}\int dx \mathcal{G}(x,x,E) &= \int dx \mathcal{G}_0(x,x,E) - \frac{\lambda}{1+\lambda\mathcal{G}_0(a,a,E)}\frac{\partial}{\partial E}\mathcal{G}_0(a,a,E) \\ &= \int dx \mathcal{G}_0(x,x,E) - \frac{\partial}{\partial E}\ln\left(1+\lambda\mathcal{G}_0(a,a,E)\right)\,.\end{aligned} \tag{2.10}$$

Substituting into eq. (2.5), we find

$$\mathcal{E} = -\frac{1}{2\pi}\,\mathrm{Im}\int_0^\Omega dE\sqrt{E}\,\frac{\partial}{\partial E}\ln\left(1+\lambda\mathcal{G}_0(a,a,E+i\epsilon)\right) \tag{2.11}$$

where we have dropped a term of the form $\mathcal{E}_0 = \frac{1}{2\pi}\,\mathrm{Im}\int_0^\Omega dE\sqrt{E}\int dx\mathcal{G}_0(x,x,E+i\epsilon)$, which does not depend on a and therefore does not contribute to the Casimir force, $-d\mathcal{E}/da$. Since we are only interested in the force, not the energy, we differentiate with respect to a, $\mathcal{F} = -\partial\mathcal{E}/\partial a$,

$$\mathcal{F} = \frac{1}{2\pi}\,\mathrm{Im}\int_0^\infty dE\sqrt{E}\,\frac{\partial^2}{\partial E\partial a}\ln\left(1+\lambda\mathcal{G}_0(a,a,E+i\epsilon)\right) \tag{2.12}$$

where we have taken $\Omega \to \infty$ since according to our previous discussions the limit exists and is finite.

The analytic structure of the integrand of eq. (2.12) is important for our analysis (see figure 2) because only the imaginary part contributes to $\mathcal{F}$, and the integrand is real for real E except at its singularities. The $\sqrt{E}$ gives a branch cut running from $E=0$ to ∞, which we place along the negative real axis. For real, positive E there are two regions of interest. Above threshold for scattering, $E > m^2$, $\mathcal{G}_0$ is complex, so the integrand has a cut with branch point at $E = m^2$. For E real and below threshold $\mathcal{G}_0(a,a,E)$ is real, so the only singularities occur when the argument of the logarithm, $1+\lambda\mathcal{G}_0(a,a,E+i\epsilon)$, vanishes, where the integrand has poles. When E is below the spectrum of $\mathcal{H}_0$, $\mathcal{G}_0(a,a,E)$ is positive and $1+\lambda\mathcal{G}_0(a,a,E)$ cannot vanish. So there are no singularities in the domain

[7] After this paper was completed Brian Winn, in a conversation with one of us (AS), pointed out that singularly perturbed hamiltonians like (2.3) have been studied in chaotic billiards theory where they are called *Šeba billiards* and generalizations of eq. (2.8) can be found in the literature on this subject [7].

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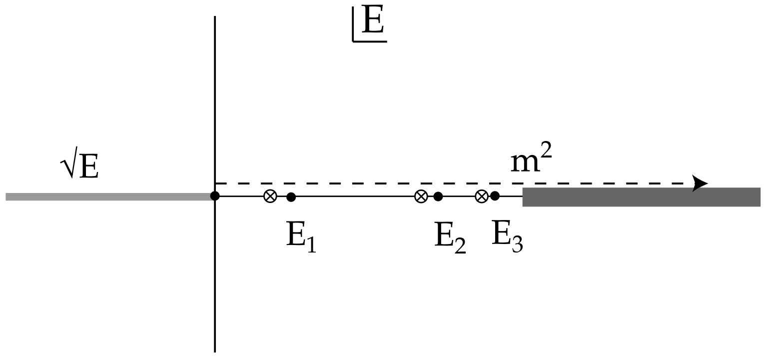

Figure 2: Analytic structure of the integrand of eq. (2.12) in the complex E-plane. The left hand cut comes from the $\sqrt{E}$, the right hand cut beginning at $E = m^2$ comes from scattering states. The pole contributions are marked by solid circles. They lie just above the bound states of $V(x)$, marked with $\otimes$.

$0 < E < m^2$ unless $\mathcal{H}_0$ has bound states. Suppose, then, that $\mathcal{H}_0$ has bound states at E_1, E_2, E_3, $\ldots E_M$. Because $\mathcal{G}_0 \sim 1/(E_j - E)$ near the j^{th} bound state, is easy to see that $1+\lambda\mathcal{G}_0(a,a,E)$ must vanish at a value of E between the j^{th} and $j+1^{\text{th}}$ bound states of $\mathcal{H}_0$. At each of these energies there is a contribution to the imaginary part from the $i\epsilon$ in the argument of $\mathcal{G}_0$. Therefore at least $M-1$ poles contribute to $\mathcal{F}$. If the pole just above E_M occurs at an energy below m^2 then there is one more. These contributions have a simple physical interpretation: They are the contributions to the Casimir force from the bound states in $V(x)$ subject to the boundary condition $\Delta\psi'(a) = \lambda\psi(a)$. The analytic structure of the integrand of eq. (2.12) in the complex E-plane is summarized in figure 2.

From these considerations it is clear that the problem is simplified if we rotate the integration contour to the negative imaginary axis. Making the obvious analogy to Feynman diagram methods, we refer to this as “Wick rotation” to the “euclidean form” of the Casimir buoyancy. There is no contribution to the force from the semi-circle at large $|E|$, because for $E \gg V$ we have $\mathcal{G}_0(a,a,E) \sim \frac{i}{2\sqrt{E}}$. Although this yields a contribution to $\mathcal{E}$ (logarithmically divergent in the cutoff Ω), it is independent of a, and therefore does not affect the force.

The result,

$$\mathcal{F} = \frac{1}{2\pi}\int_0^\infty dE\sqrt{E}\,\frac{\partial^2}{\partial E\partial a}\ln\left(1+\lambda\mathcal{G}_0(a,a,-E)\right), \tag{2.13}$$

is particularly useful because the argument of the logarithm is positive definite. Eq. (2.13) can be integrated by parts without contributions at the limits because the surface term at $E\to\infty$ is independent of a and the surface term at $E=0$ vanishes:

$$\mathcal{F}(a,\lambda,V) = -\frac{1}{4\pi}\int_0^\infty dE\frac{1}{\sqrt{E}}\frac{\partial}{\partial a}\ln\left(1+\lambda\mathcal{G}_0(a,a,-E)\right). \tag{2.14}$$

Here we have restored some of the arguments on the function $\mathcal{F}$ as a reminder of its important variation with position, λ, and background field, V. It will be sometimes convenient to introduce the imaginary momentum $\kappa = \sqrt{E+m^2}$ (here E is the dummy variable in (2.14)) in the euclidean domain,

$$\mathcal{F}(a,\lambda,V) = -\frac{1}{2\pi}\int_m^\infty d\kappa\kappa\frac{1}{\sqrt{\kappa^2-m^2}}\frac{\partial}{\partial a}\ln\left(1+\lambda\mathcal{G}_0(a,a,-\sqrt{\kappa^2-m^2})\right). \tag{2.15}$$

This result can be rewritten usefully by introducing the Jost solutions to the Schrödinger equation, $\mathcal{H}\psi = -E\psi$ (notice the minus sign in front of E, due to the Wick rotation), which are defined by their behavior as $x \to \pm\infty$ [8],

$$\begin{aligned}\lim_{x\to\infty}\psi^+(\kappa,x)e^{\kappa x} &= 1\\ \lim_{x\to-\infty}\psi^-(\kappa,x)e^{-\kappa x} &= 1\end{aligned} \tag{2.16}$$

The boundary conditions of eqs. (2.16) render the functions $\psi^\pm$ analytic for $\mathrm{Re}\,\kappa > 0$. On the contour $-\infty < E < 0$ needed in eq. (2.15) the Jost solutions are real and fall exponentially in the limits given in eq. (2.16). The Greens function can be written in terms of the Jost solutions,

$$\mathcal{G}_0(x,y,-\sqrt{\kappa^2-m^2}) = \frac{T(\kappa)}{2\kappa}\psi^+(\kappa,x_>)\psi^-(\kappa,x_<) \tag{2.17}$$

where $x_>$ ($x_<$) is the greater (lesser) of x or y, and $T(\kappa)$ is the transmission coefficient. Upon substituting for $\mathcal{G}_0$ in eq. (2.15), we obtain

$$\mathcal{F}(a,\lambda,V) = -\frac{\hbar c}{2\pi}\int_0^\infty d\kappa\kappa\frac{1}{\sqrt{\kappa^2-m^2}}\frac{\partial}{\partial a}\ln\left(1+\frac{\lambda T(\kappa)}{2\kappa}\psi^+(\kappa,a)\psi^-(\kappa,a)\right) \tag{2.18}$$

In the last equation we have restored the factors of $\hbar$ and c ($E = \kappa^2 - m^2$ has units of $1/\text{length}^2$ in these units).

2.3 $\lambda \to \infty$: the Dirichlet case

When the strength, λ, of the sharp background field becomes large compared to the eigenvalue, ω, the mode of the field with frequency ω vanishes at $x = a$. If all the modes that contribute to the buoyancy have eigenvalues below λ, then effectively the field itself obeys the boundary condition $\phi(a,t) = 0$. As discussed in ref. [5], this description is acceptable if the physical observable of interest remains becomes independent of λ in the limit. In the case of buoyancy force in one dimension the limit $\lambda \to \infty$ exists,

$$\lim_{\lambda\to\infty}\mathcal{F}(a,\lambda,V) \equiv \mathcal{F}_\infty(a,V) = -\frac{1}{2\pi}\int_0^\infty\frac{dE}{\sqrt{E}}\frac{\partial}{\partial a}\ln\mathcal{G}_0(a,a,-E) \tag{2.19}$$

which can be also written as

$$\mathcal{F}_\infty(a,V) = -\frac{1}{4\pi}\int_m^\infty d\kappa\kappa\frac{1}{\sqrt{\kappa^2-m^2}}\left(\frac{1}{\psi^+(\kappa,a)}\frac{\partial\psi^+(\kappa,a)}{\partial a}+\frac{1}{\psi^-(\kappa,a)}\frac{\partial\psi^-(\kappa,a)}{\partial a}\right). \tag{2.20}$$

It is instructive to study the contribution to the Casimir force of the bound states in the $\lambda \to \infty$ limit.[8] Returning to the non-rotated formula eq. (2.12), taking $\lambda \to \infty$ and

[8]The analysis for finite λ is similar, but technically more complicated. In practice, following the discussion in the previous section, one has to find the poles of $1+\lambda\,\mathcal{G}_0(a,a,E)$ say $E(a)$, which are also eigenvalues of $\mathcal{H}$, and proceed as before.

substituting the Jost solutions representation for the Greens function, we have

$$\mathcal{F}_\infty(a,V) = -\frac{1}{2\pi}\,\mathrm{Im}\int_0^\infty dE\frac{1}{\sqrt{E}}\frac{\partial}{\partial a}\ln\left(\psi^+(E,a)\psi^-(E,a)\right). \tag{2.21}$$

Remembering that Jost solutions are real for $E < m^2$, it is clear that the only contributions in this range come from possible zeros in $\psi^\pm(E,a)$, where the integrand picks up an imaginary part because the integration contour goes above the pole on the $\mathrm{Re}\,E$ axis. The locations of these zeros depends on a, so we label them as $E_{j^\pm}(a)$. Near the j^{th} zero in $\psi^\pm$,

$$\psi^\pm(E,a) = (E - E_{j^\pm}(a) + i\epsilon)\left.\frac{\partial\psi^\pm(E,a)}{\partial E}\right|_{E=E_{j^\pm}(a)} + \cdots \tag{2.22}$$

where we have restored the $i\epsilon$ to make the nature of the singularity clear. Substituting into eq. (2.21) and extracting the imaginary part, we obtain a very simple expression for the bound state contribution to $\mathcal{F}_\infty$,

$$\mathcal{F}_\infty(a,V)|_{\text{bound states}} = -\frac{\hbar c}{4}\sum_{j^\pm}\frac{1}{\sqrt{E_{j^\pm}(a)}}\frac{\partial E_{j^\pm}(a)}{\partial a} = -\frac{\partial}{\partial a}\left[\frac{\hbar c}{2}\sum_{j^\pm}\sqrt{E_{j^\pm}(a)}\right]. \tag{2.23}$$

Here we have used the fact that the various partial derivatives are related by

$$d\psi = \frac{\partial\psi}{\partial E}dE + \frac{\partial\psi}{\partial a}da = 0 \tag{2.24}$$

along the contour $\psi^\pm(E_{j^\pm}(a),a) = 0$ in the E, a plane.

The simple form of eq. (2.23) has a simple origin: the $\{\sqrt{E_{j^\pm}(a)}\}$ are the eigenenergies of ϕ in $V(x)$ constrained to vanish at $x = a$, so the quantity in brackets is the bound state contribution to the Casimir energy for a scalar field in the potential $V(x)$ *forced to vanish at* $x = a$. This is an equivalent formulation of the Casimir buoyancy in the Dirichlet ($\lambda \to \infty$) limit. From this interpretation it is clear that the number of terms in the sum in eq. (2.23) lies between $N - 1$ and N, where N is the number of bound states in $V(x)$.

To complete this parametrization of the Casimir buoyancy we write the continuum contribution as an integral over the scattering momentum k, so another expression for the total buoyancy in the $\lambda \to \infty$ limit, equivalent to eq. (2.20) is

$$\mathcal{F}_\infty(a,V) = -\frac{\partial}{\partial a}\left[\frac{\hbar c}{2}\sum_{j^\pm}\sqrt{E_{j^\pm}(a)} - \frac{\hbar c}{2\pi}\,\mathrm{Im}\int_0^\infty dk\frac{k}{\sqrt{k^2+m^2}}\ln\left(\psi^+(E,a)\psi^-(E,a)\right)\right]. \tag{2.25}$$

If the k-integration were rotated to the positive imaginary axis, $k \to i\kappa$, two kinds of contributions would arise: (a) from the cut from $\frac{1}{\sqrt{k^2+m^2}}$ and (b) from poles in $\psi^+\psi^-$ corresponding to the same states counted in the sum over j. The pole contribution would exactly cancel the sum and the integral over κ would yield eq. (2.20) as it must.

3. Approximations and special cases

Casimir buoyancy is an unfamiliar phenomenon. Its properties are not readily apparent from the general expressions, eqs. (2.20) and (2.25). In this section we study the buoyancy in special situations where it simplifies. First we look at smooth, deep potentials where the WKB approximation applies. Next we look at reflectionless potentials, where we show that only the bound states contribute to the buoyancy. Third, we look at the buoyancy in the domain beyond the range of $V(x)$, and finally we study the first Born approximation, where the buoyancy is simple, but not local.

3.1 WKB approximation

If the potential is smooth and strong, a WKB expansion can be made for the Casimir buoyancy. We work with the Wick rotated form, eq. (2.20). To apply WKB [9] we introduce a fictitious "$\tilde{\hbar}$" into the hamiltonian, $\mathcal{H} \to -\tilde{\hbar}^2 \frac{d^2}{dx^2} + W(x)$ (henceforth we will write $W(x)$ for $m^2 + V(x)$) and write

$$\psi^{\pm}(E,x) = \exp\left(-\frac{1}{\tilde{\hbar}} s_0(x,E) + s_1(x,E) + \cdots\right). \tag{3.1}$$

and find

$$\begin{aligned} s_0(x,E) &= \pm \int_0^x dy \sqrt{-E + W(y)} \\ s_1(x,E) &= -\frac{1}{4} \ln(-E + W(x)) \end{aligned} \tag{3.2}$$

so the domain $E < 0$ corresponds to the WKB forbidden region (since $W(x) > 0$, which we have assumed from the outset) and gives real $s_{0,1,\ldots}$, $\psi_\pm$ and $\mathcal{G}_0$.

The criterion for the validity of the WKB expansion (with $\tilde{\hbar}$ set to unity), $s_1' \ll s_0'$, reduces to

$$\frac{d}{dx}\frac{1}{\sqrt{W(x)}} \ll 1\,, \tag{3.3}$$

from which we conclude that $W'(x)/W^{3/2}$ should be small and in particular where $W = m^2 + V(x)$ should not vanish[9] (i.e. at turning points). Hence the WKB approximation applies to potentials which are "deep" in the sense that $\int dy \sqrt{W(y)} \gg 1$ and far from turning points where $W(x) = 0$.

It is straightforward to construct the Greens function, in the WKB approximation,

$$\mathcal{G}_{0\ \mathrm{WKB}}(a,a,-E) = \frac{1}{2\sqrt{E + W(a)}} \tag{3.4}$$

and, upon substituting into eq. (2.14), and performing the integral over E, we find

$$\begin{aligned} \mathcal{F}(\lambda,a)_{\mathrm{WKB}} &= \frac{\partial}{\partial a}\frac{\hbar c}{4\pi}\left(\pi\sqrt{W(a)} + \lambda \ln(W(a)) - \sqrt{\lambda^2 - 4W(a)}\,\ln\frac{\lambda - \sqrt{\lambda^2 - 4W(a)}}{\lambda + \sqrt{\lambda^2 - 4W(a)}}\right) \\ &\equiv -\frac{\partial}{\partial a}\Omega(a,\lambda)_{\mathrm{WKB}}\,, \end{aligned} \tag{3.5}$$

is a local function of the potential, $V(a)$. We call this function the *quantum potential* $\Omega(a,\lambda)$.

[9] If $W \sim x^{2n}$ then $W'/W^{3/2} \sim x^{-n-1}$ becomes arbitrarily large as $x \to 0$.

Equation (3.5) simplifies considerably in the Dirichlet limit,

$$\mathcal{F}_{\infty\ \mathrm{WKB}}(a) = \frac{\partial}{\partial a}\frac{\hbar c}{4}\sqrt{m^2 + V(a)} \equiv -\frac{\partial}{\partial a}\Omega(a)_{\mathrm{WKB}}\,. \tag{3.6}$$

A remarkably simple result for such a complex phenomenon. One general feature can be easily deduced from this form of the quantum potential Ω: it decreases when the potential $V(x)$ increases and hence the force felt by the plate is a *buoyancy* force. We will encounter this phenomenon throughout the rest of this paper in extensions (to regions where the simple WKB form is not valid, to non-zero temperature and to higher number of dimensions) and several examples.

3.2 Points at which $V(x) + m^2 = 0$

When the potential is smooth (with a definite length-scale $b \sim [\frac{1}{W}\frac{dW}{dx}]^{-1}$) but there exists an x_0 such that $V(x_0) + m^2 = 0$ (or in general an interval where $(V(x) + m^2)b^2 \ll 1$) then WKB cannot be applied. The simple formula eq. (3.6) breaks down. For example in the case of symmetric potential $m^2 + V(x) = x^2$ the force must vanish at $x = 0$ for symmetry, while eq. (3.6) predicts $\mathcal{F} \to$ constant.

To understand what really happens when the delta function reaches the point x_0 one needs a more clever guess than WKB. Moreover, adding more terms of the WKB series cannot help since the asymptotic nature of WKB means that adding terms improves the result for *larger* values of $b^2(V + m^2)$, so certainly not close to x_0. The formally correct procedure would be to find a differential equation 'similar' to the one we are studying but solvable and a smooth map from one to the other [10]. In this way we would obtain a *uniform* approximation near the turning point x_0. However such an analysis goes beyond the goal of this paper, so to clarify the situation we will assume that close to $x_0 = 0$ we have (remember that $W(x) \geq 0$)

$$W(x) = V(x) + m^2 \simeq \omega^2 x^2 + \mathcal{O}\left(x^3\right). \tag{3.7}$$

If we have $0 \neq W(0) \equiv W_0 \ll 1/b^2$, this constant could be reabsorbed by a shift of E and the range of integration in eq. (2.24), we will consider this case at the end of this section.

We then study the propagator in the neighborhood of $x_0 = 0$ i.e. the propagator $\mathcal{G}_0$ of the equation

$$-\phi''(x) + \omega^2 x^2 \phi(x) = E\phi(x) \tag{3.8}$$

which is the usual harmonic oscillator problem. We now set $\omega = 1$ and we will reintroduce it only at the end of the calculation to have dimensionally correct results.

From the two independent Jost solutions of this equation it is straightforward to write the the propagator for the harmonic oscillator as

$$\mathcal{G}_0(a, a, E) = \frac{2^{-(E+1)/2}}{\sqrt{\pi}}\Gamma\left(\frac{1-E}{2}\right) e^{-a^2} H_{\frac{E-1}{2}}(a) H_{\frac{E-1}{2}}(-a) \tag{3.9}$$

where H is the Hermite function, a generalization of the Hermite polynomials related to the parabolic cylinder function [11]. Notice that the poles of the gamma function give the correct spectrum of the bound states.

One can then calculate the buoyancy force for any λ, by inserting eq. (3.9) into eq. (2.14) and performing the integral numerically. In the Dirichlet case $\lambda \to \infty$ however the results simplifies using eq. (2.19) to

$$\mathcal{F}_{\infty\ \text{h.o.}} = -\frac{1}{4\pi}\int_0^\infty \frac{dE}{\sqrt{E}}\left((1+E)\left(\frac{H_{-(3+E)/2}(-a)}{H_{-(1+E)/2}(-a)} - \frac{H_{-(3+E)/2}(a)}{H_{-(1+E)/2}(a)}\right) - 2a\right). \tag{3.10}$$

In this case the expression for small a can be recovered by expanding the integrand in powers of a and integrating term by term (the expansion can be carried on to any order of a because the integrals over E converge):

$$\begin{aligned}\mathcal{F}_{\infty\ \text{h.o.}} &\simeq \frac{1}{4\pi}\left(a\left(\int_0^\infty dE\frac{1}{\sqrt{E}}\left(\frac{8\,\Gamma(\frac{3+E}{4})^2}{\Gamma(\frac{1+E}{4})^2} - 2E\right)\right) + \mathcal{O}\left(a^3\right)\right)\\ &\simeq \frac{1}{4\pi}\omega^{3/2}a(3.24\ldots) + \mathcal{O}\left(\omega^{5/2}a^3\right)\end{aligned} \tag{3.11}$$

which again exhibits the buoyancy phenomenon, i.e. the Casimir force pushes the plate toward points at higher potential. Notice that the force vanishes as $a \to x_0 = 0$ as it should from symmetry arguments.[10]

If $V(x_0) + m^2 = W_0$ and $W_0 \ll 1/b^2$ ($b^2 = 1/\omega$ in this case) one can repeat the preceding derivation, change the lower limit of integration over E from 0 to $W_0 b^2$ (W_0/ω in the case at hand) and write

$$\mathcal{F}_{\infty\ \text{h.o.}} \simeq \frac{1}{4\pi}\omega^{3/2} a f\left(\frac{W_0}{\omega}\right), \tag{3.12}$$

where we have defined

$$f(x) = \int_x^\infty dy \frac{1}{\sqrt{y-x}}\left(\frac{8\,\Gamma(\frac{3+y}{4})^2}{\Gamma(\frac{1+y}{4})^2} - 2y\right), \tag{3.13}$$

and $f(0) = 3.24\ldots$ Although we do not have an analytic expression for f it can be studied numerically with ease and to a great accuracy. In the opposite limit $W_0/\omega \gg 1$ one can still shift the range of integration and then, by working out the asymptotic limit of the propagator eq. (3.9) as $E/\omega \gg 1$, one finds the now familiar WKB result eq. (3.6). We will apply eq. (3.12) to the case of Pöschl-Teller potential in section 4.2.

One can also study higher order zeros (or minima) of W, i.e. points where $W(x) \sim x^{2n}$ in a similar fashion. The Jost solutions of such a problem should be found, possibly in a series for small x, and by means of them the propagator can be written explicitly in the neighborhood of the minimum $x = 0$. Here we will not extend our analysis to those cases since no qualitatively new phenomenon arises.

[10] If we want to compare this result with an exact one we can choose $n = 4$ Poschl-Teller (see section 4.2) with $m = \sqrt{20}$ for which $V(x) + m^2 \sim 20x^2$ near $x = 0$. In this case $\mathcal{F}_{\text{exact}}/\mathcal{F}_{\text{approx}} = 0.94$ as $a \ll 1$. Such an agreement must be considered impressive, since the propagator is not a local function of the potential V and a local approximation to V does not guarantee at all a local approximation to G. Evidently, for sufficiently smooth potentials this turns out to be the case.

3.3 Temperature dependence in the WKB approximation

One can also study the case where the field is in a non-zero temperature thermal state rather than a vacuum state. The Casimir energy is then the integral of the energy momentum tensor component T_{00} evaluated on the thermal state. The result is [12]

$$\mathcal{E}_\lambda = -\frac{1}{2\pi}\,\mathrm{Im}\int_0^\Omega dE\sqrt{E}\,\coth\left(\beta\frac{\sqrt{E}}{2}\right)\frac{\partial}{\partial E}\log\left(1+\lambda\mathcal{G}_0(a,a,E)\right). \tag{3.14}$$

One can Wick-rotate this expression as well, since the Matsubara poles are on the negative real E axis. After the rotation the Greens function becomes real (the i from $\sqrt{E}$ cancels with an i from coth) and one has an imaginary part from the cot function. Dropping a term $\propto \lambda\log\Omega$ arising from the semicircle at $|E| = \Omega$ one has

$$\mathcal{E}_\lambda = -\frac{1}{2\pi}\int_0^\Omega dE\sqrt{E}\,\mathrm{Im}\left\{\cot\left(\beta\frac{\sqrt{E}}{2}-i\epsilon\right)\right\}\frac{\partial}{\partial E}\log\left(1+\lambda\mathcal{G}_0(a,a,-E)\right). \tag{3.15}$$

In the limit $\epsilon \to 0$ one has $\mathrm{Im}\cot(x-i\epsilon) = \pi\sum_n \delta(x-n\pi)$. The final expression is then a sum over Matsubara frequencies.

$$\mathcal{E}_\lambda = -\frac{1}{2}\int_0^\Omega dE\sqrt{E}\sum_n\delta\left(\beta\frac{\sqrt{E}}{2}-n\pi\right)\frac{\partial}{\partial E}\log\left(1+\lambda\mathcal{G}_0(a,a,-E)\right). \tag{3.16}$$

The study of this expression would require a paper on its own. Here we will limit our analysis of the WKB case, since a closed exact expression can be easily obtained in this case. To obtain it however, we have to go back to the original, non Wick-rotated WKB expression eq. (3.14) (we again use the notation $W(x) = V(x) + m^2 > 0$)

$$\mathcal{E}_\lambda \simeq -\frac{1}{2\pi}\,\mathrm{Im}\int_0^\Omega dE\sqrt{E}\,\coth\left(\beta\frac{\sqrt{E}}{2}\right)\frac{\partial}{\partial E}\log\left(1+\frac{i\lambda}{2\sqrt{E+i\epsilon-W(a)}}\right). \tag{3.17}$$

The force in the Dirichlet limit (the cutoff Ω can again be sent to ∞ without problems) is

$$\mathcal{F}_\infty \simeq \frac{1}{4\pi}\frac{dV}{da}\,\mathrm{Im}\int_0^\infty dE\sqrt{E}\,\coth\left(\beta\frac{\sqrt{E}}{2}\right)\frac{\partial}{\partial E}\left(\frac{1}{E-W(a)+i\epsilon}\right), \tag{3.18}$$

and by trading the E derivative with a $W(a)$ derivative inside the integral, taking the imaginary part and the limit $\epsilon \to 0$ we find

$$\begin{aligned}\mathcal{F}_\infty(a,\beta) &\simeq -\frac{1}{4\pi}\frac{dW}{da}\frac{\partial}{\partial W}\int_0^\infty dE\sqrt{E}\,\coth\left(\beta\frac{\sqrt{E}}{2}\right)\left(-\pi\delta(E-W)\right)\\ &= \frac{1}{4}\frac{d}{da}\left(\sqrt{W}\,\coth\left(\frac{\beta\sqrt{W}}{2}\right)\right) \equiv -\frac{d}{da}\Omega(a,\beta)\,,\end{aligned} \tag{3.19}$$

where the last expression defines again a local *thermal quantum potential*

$$\Omega(a,\beta) = -\frac{1}{4}\sqrt{W}\,\coth\left(\frac{\beta\sqrt{W}}{2}\right). \tag{3.20}$$

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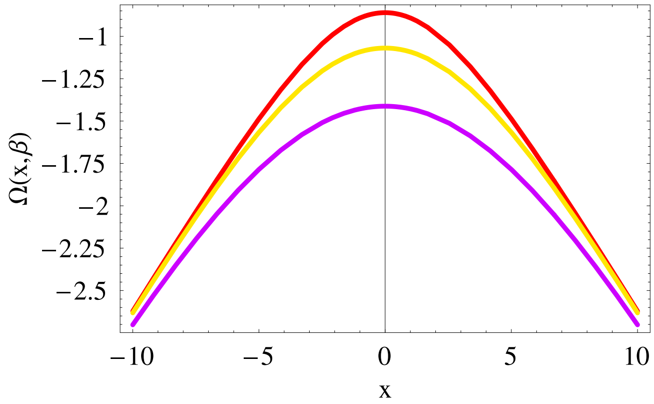

Figure 3: Turning on the temperature does not affect the Casimir buoyancy qualitatively. Here we plot Ω for the potential $W(x) = 10 + x^2$ and $\beta = 1, 0.6, 0.4$ from up down respectively. The minimum of W at $x = 0$ is always a maximum of Ω_{WKB}.

In the low temperature limit $T \to 0$ we have $\Omega \to -\sqrt{W}/4$ as it should (compare with eq. (3.6)). In the high temperature limit $T \to \infty$, $\beta \to 0$ one finds $\mathcal{F}_\infty \propto T^{-1} dW/da$. Contrary to what one could expect from the known 'classical limit' $\mathcal{F} \propto T/a$ of Casimir force between rigid bodies [13, 12], the force goes to zero when the temperature grows indefinitely. However here we are in a totally different limit, where the background potential is slowly varying and we are considering only the zero reflection term for $\mathcal{G}_0$. The rigid bodies expansion of $\mathcal{G}$ that gives rise to the well-known Casimir force actually is made up of non-local reflection contributions [14].

This expression is valid whenever WKB is valid, hence when the length-scale of the potential b is such that $Wb^2 \gg 1$. We have already discussed the buoyancy effect, i.e. the quantum potential has a maximum where V has a minimum. A non-zero temperature does not modify this prediction qualitatively as can be seen from figure 3.

3.4 Reflectionless potentials

The continuum contribution to the Casimir buoyancy vanishes in the Dirichlet limit if a potential is reflectionless. For a reflectionless potential, $\psi^+(k,x) \to \frac{1}{T(k)} e^{ikx}$ as $x \to -\infty$, so $\psi^-(k,x) = \frac{1}{T^*(k)} \psi^{+*}(k,x)$. In this case $\psi^+(k,a)\psi^-(k,a) = |\psi^+(k,a)|^2 / T^*(k)$. Looking back to eq. (2.25), we see that the imaginary part of the logarithm is independent of a so the force vanishes. Thus the Casimir buoyancy in a reflectionless potential is entirely determined by the bound states.

3.5 Beyond the range of the potential

Here we consider a background potential which vanishes identically for $|x| > b$. The same results apply to a short range potential ($V(x) \propto e^{-\mu|x|}$) in the limit $x \to \infty$. For $x < -b$ in the euclidean domain,

$$\begin{aligned} \psi^-(\kappa, x) &= e^{\kappa x} \\ \psi^+(\kappa, x) &= \frac{1}{T(\kappa)} e^{-\kappa x} + \frac{R(\kappa)}{T(\kappa)} e^{\kappa x} \end{aligned} \tag{3.21}$$

The Greens function becomes

$$\mathcal{G}_0(a,a,-\sqrt{\kappa^2-m^2}) = \frac{1}{2\kappa}\left(1+R(\kappa)e^{2\kappa a}\right) \tag{3.22}$$

so the buoyancy can be written,

$$\mathcal{F}(\lambda,a) = -\frac{\hbar c}{2\pi}\int_m^\infty \frac{\kappa d\kappa}{\sqrt{\kappa^2-m^2}}\frac{\partial}{\partial a}\ln\left(1+\frac{\lambda}{2\kappa}\left(1+R(\kappa)e^{2\kappa a}\right)\right) \qquad \text{for } a<-b\,. \tag{3.23}$$

For $x>b$ a similar expression holds with $R\to\overline{R}$, the reflection coefficient on the right. If the potential is symmetric, then $\overline{R}=R$. If the mass, m, of the fluctuating field is non-zero, then $\mathcal{F}$ falls like e^{-2ma} when $a>b$. The explicit form depends on the reflection coefficient in the euclidean domain. In section 4 the explicit example of a δ-function background is studied.

3.6 First Born approximation

When the background potential is weak the Casimir buoyancy can be expanded in powers of V. The first term is simple when $m=0$ and $\lambda\to\infty$. A straightforward calculation gives the Greens function to $\mathcal{O}(V)$,

$$\mathcal{G}_0(a,a,E) = \frac{i}{2k}+\frac{1}{4k^2}\left(e^{2ika}\int_{-\infty}^a dy e^{-2iky}V(y)+e^{-2ika}\int_a^\infty dy e^{2iky}V(y)\right)+\mathcal{O}(V^2)\,. \tag{3.24}$$

$\mathcal{G}_0$ has no bound states at this order, so only the continuum contribution in eq. (2.25) need be calculated. Straightforward evaluation leads to

$$\mathcal{F}_1 = -\frac{\hbar c}{2\pi}\int_{-\infty}^\infty \frac{dz}{2z}V(z+a) = -\frac{\hbar c}{4}\mathbf{H}[V,a] \tag{3.25}$$

(where the integral is intended as the Cauchy principal value) which is a non-local functional of V known to mathematicians as the *Hilbert transform* $\mathbf{H}$ of V [15] at the point a.

4. Examples

In this section we report the Casimir buoyancy in three explicit sample background potentials and use them to study the domains of application of the approximations in the previous section. First we treat the potential $\ell(\ell+1)/x^2$ on the half-line $x>0$. Second we explore the family of Pöschl-Teller potentials, $-n(n+1)\,\mathrm{sech}^2\,x$. Finally we study the δ-function, $\beta\delta(x)$, which has been studied in other contexts [16].

4.1 $V(x)=\ell(\ell+1)/x^2$

The potential V_0/x^2 leads to a well defined problem on the half-line $x>0$ when V_0 is positive. It is convenient to parameterize V_0 by $\ell(\ell+1)$, so we can use results familiar from the study of three dimensional central potentials. Note, however, that ℓ need not be integer. Since $V(x)$ is positive definite there is no obstruction to taking $m=0$, which we

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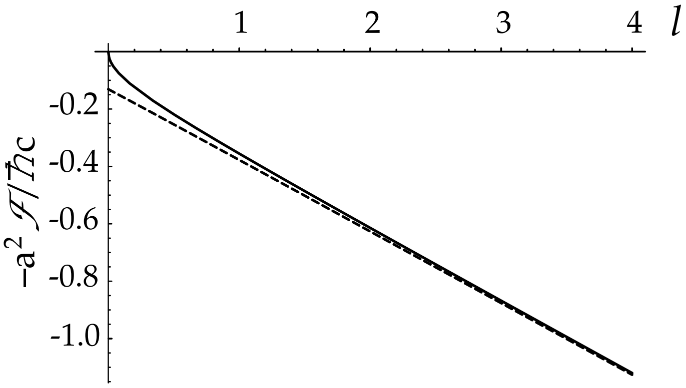

Figure 4: Casimir buoyancy, $-a^2\mathcal{F}/\hbar c$, for $V(x) = \ell(\ell+1)/x^2$ in the Dirichlet limit. The dashed curve is the exact result of eq. (4.2). The solid curve is the WKB approximation, eq. (4.3).

adopt to simplify the calculation. The formalism of section 2 has to be modified slightly on account of the boundary at $x = 0$. In particular, the Jost solution $\psi^-(x, E)$ has to be replaced by the solution, $\phi(x, E) \propto j_\ell(kx)$, *regular at* $x = 0$. The other Jost solution is given by $h_\ell^{(1)}(kr)$. With this adaptation, and noting that the potential has no bound states, we can compute the Casimir buoyancy as an integral over scattering states, rotated to imaginary momentum,

$$\mathcal{F}(\lambda, \ell, a) = -\frac{\hbar c}{2\pi a^2}\int_0^\infty d\xi \left(-\ln\left(1 + \lambda a I_{\tilde{\ell}}(\xi) K_{\tilde{\ell}}(\xi)\right) + \frac{\lambda a I_{\tilde{\ell}}(\xi) K_{\tilde{\ell}}(\xi)}{1 + \lambda a I_{\tilde{\ell}}(\xi) K_{\tilde{\ell}}(\xi)}\right) \tag{4.1}$$

where $\tilde{\ell} = \ell + 1/2$, and I_ν and K_ν are modified Bessel functions. It is easy to verify that the ξ integral is convergent. In the Dirichlet limit the buoyancy is simply proportional to $1/a^2$ (for dimensional reasons it could not be different),

$$\lim_{\lambda\to\infty} \mathcal{F}(\lambda, \ell, a) = \mathcal{F}_\infty(\ell, a) = -\frac{\hbar c}{2\pi a^2}\int_0^\infty d\xi \left(1 + \xi\frac{I'_{\tilde{\ell}}(\xi)}{I_{\tilde{\ell}}(\xi)} + \xi\frac{K'_{\tilde{\ell}}(\xi)}{K_{\tilde{\ell}}(\xi)}\right) \tag{4.2}$$

The potential $\ell(\ell+1)/x^2$ is neither weak nor reflectionless. It is however smooth and deep, so the WKB approximation should be accurate. $\mathcal{F}_{\text{WKB}}(\lambda, \ell, a)$ can be calculated easily from eq. (3.6), and the limit $\lambda \to \infty$ is particularly simple,

$$\mathcal{F}_{\text{WKB}\ \infty}(\ell, a) = -\frac{\hbar c\sqrt{\ell(\ell+1)}}{4a^2}\,. \tag{4.3}$$

The exact result and the WKB approximation are compared in the $\lambda \to \infty$ limit in figure 4. The difference is very small once $\ell \gtrsim 1$. In fact WKB does even better: if we make the standard Langer replacement, $\sqrt{\ell(\ell+1)} \to (\ell+\frac{1}{2})$ [9, 10], then the WKB and exact results coincide within the widths of the lines in figure 4.

In figure 5 we compare the exact result with the WKB approximation for finite λ. Clearly WKB is an excellent approximation in this case.

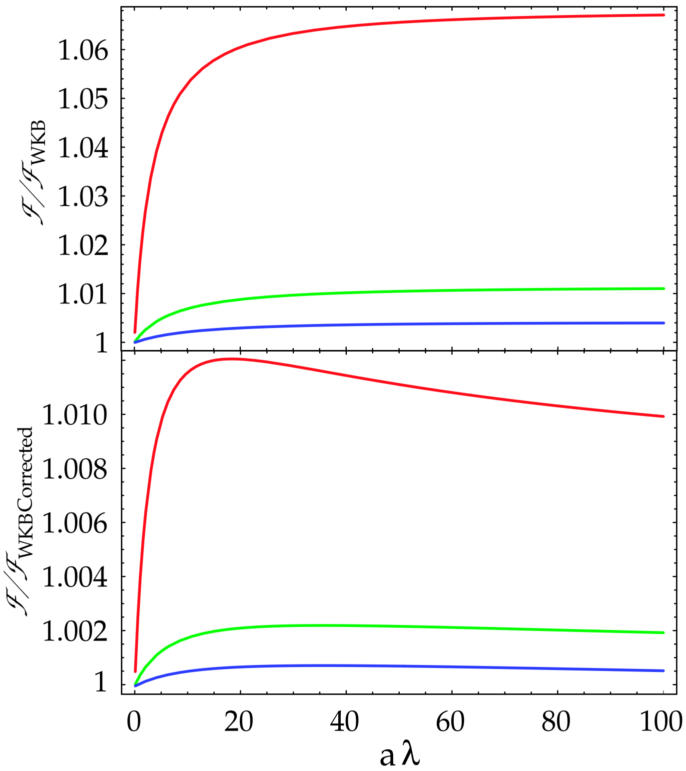

Figure 5: (a) Ratio of Casimir buoyancy to the WKB approximation for $V(x) = \ell(\ell+1)/x^2$. Up to down (red to blue) $\ell = 1, 3, 5$; (b) Same as (a) with the "correction", $\ell(\ell+1) \to (\ell+\frac{1}{2})^2$.

4.2 Pöschl-Teller potentials

The potentials $V_n^{\text{PT}}(x) = -n(n+1)\,\text{sech}^2(x)$, $n = 1, 2, 3, \ldots$, are reflectionless and the associated solutions to the Schrödinger equation can be expressed in terms of elementary functions.[11] The Casimir buoyancy can be computed from the bound states alone. For simplicity we restrict ourselves to the $\lambda \to \infty$ limit in this case. In practice, it is easier to compute the integrals in the euclidean region using eq. (2.20) since the scattering wave-functions, $\psi^{\pm}(E, a)$ are easy to construct but the roots of $\psi^{\pm}(E_{j^{\pm}}(a), a) = 0$ are hard to find for $n > 2$. The necessary Jost solutions for imaginary momentum are given by

$$\psi_n^{\pm}(\kappa, x) = \left(\frac{d}{dx} - \tanh x\right)\left(\frac{d}{dx} - 2\tanh x\right)\cdots\left(\frac{d}{dx} - n\tanh x\right) e^{ikx}\bigg|_{k=\pm i\kappa} \tag{4.4}$$

up to an inessential normalization. The necessary product of Jost solutions is given by

$$\begin{aligned}
\psi_1^{+}(\kappa, a)\psi_1^{-}(\kappa, a) &= \tanh^2 a - \kappa^2 \qquad (4.5)\\
\psi_2^{+}(\kappa, a)\psi_2^{-}(\kappa, a) &= (1 - \kappa^2 - 3\kappa\tanh a - 3\tanh^2 a)(1 - \kappa^2 + 3\kappa\tanh a - 3\tanh^2 a)
\end{aligned}$$

for example for $n = 1$ and $n = 2$. The required integrals can be performed explicitly for $n = 1$ and numerically for larger n.

[11]The apparent dimensional inconsistency in V_n^{PT} needs a word of explanation. If we begin with a dimensionally correct hamiltonian, $-d^2/dx^2 + m^2 - V_0\,\text{sech}^2(x/b)$ and define a dimensionless unit of distance, $z = x/b$, then $V_0 b^2$ is the dimensionless potential which equals $n(n+1)$ in the Pöschl-Teller problem.

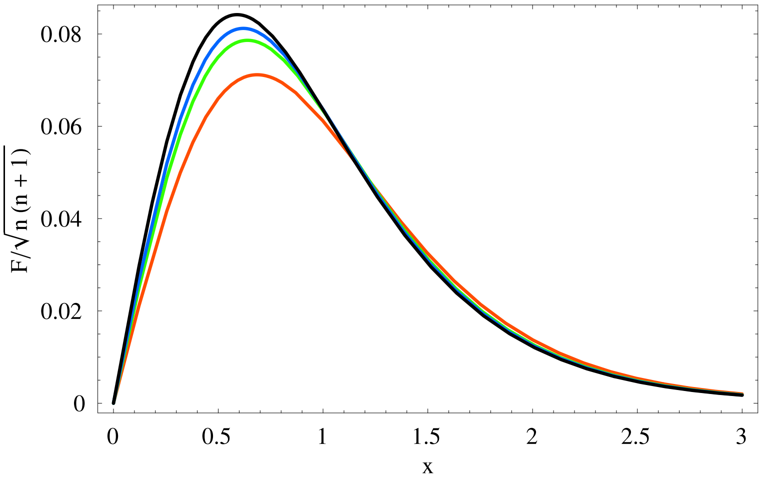

Figure 6: Casimir force for Poschl-Teller potential in the Dirichlet limit. Exact vs. WKB results. From down up (red, green and blue) we have the exact results for $n = 1, 2, 3$ and in on the top, in black, the WKB approximation. $\mu^2 \equiv m^2/n(n+1) = 2$ for all the curves.

The case $n = 1$ is particularly simple and instructive:

$$\mathcal{F}_1^{\rm PT}(m,a) = \hbar c \frac{\tanh a \,\mathrm{sech}^2 a}{2\sqrt{m^2 - \tanh^2 a}} \tag{4.6}$$

As usual the Casimir force is in the direction opposite to the gradient of $V(x)$. The WKB approximation to the Pöschl-Teller potentials is

$$\mathcal{F}_{n\ \rm WKB}^{\rm PT}(m,a) = \hbar c n(n+1) \frac{\tanh a \,\mathrm{sech}^2 a}{4\sqrt{m^2 - n(n+1)\,\mathrm{sech}^2 a}} \tag{4.7}$$

In order to understand how well the WKB result approximates the exact results we have plotted the force $\mathcal{F}$ divided by $\sqrt{n(n+1)}$ for fixed $\mu \equiv m/\sqrt{n(n+1)} = \sqrt{2}$. In these variables, the WKB result eq. (4.7) is independent of n and the the exact curves tend to the WKB curve as n is increased. This is no surprise since for large n the Poschl-Teller potential becomes more and more semiclassical.

Even for large n, however, if $\mu \simeq 1$ we cannot use WKB since the minimum of the potential at $x = 0$ is *almost* a turning point, i.e. $(V(0) + m^2)b^2 \ll 1$ (remember we set $b = 1$). In this case however we can resort to the harmonic oscillator approximation of section 3.2 which, for $a \to 0$ and specializing to the Poschl-Teller potential, takes the form (the notation is the same as in section 3.2)

$$\mathcal{F}_{n\ \rm h.o.}^{\rm PT} \simeq \frac{1}{4\pi}(n(n+1))^{3/4} a f\left(\frac{m^2 - n(n+1)}{\sqrt{n(n+1)}}\right). \tag{4.8}$$

The results are plotted in figure 7. As in the comparison with WKB the agreement is better the higher n, i.e. the more 'semiclassical' is the potential.

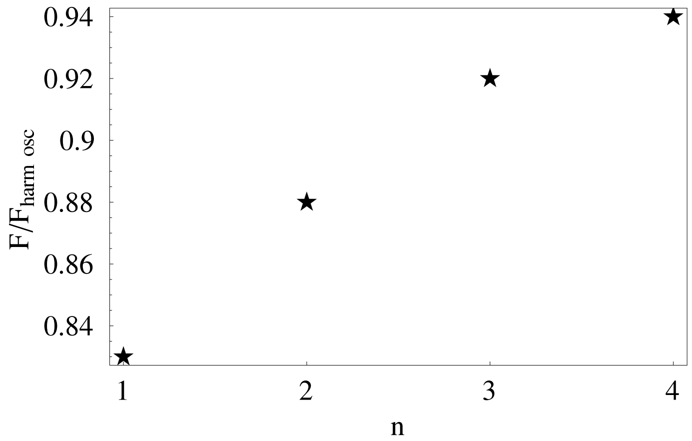

Figure 7: Casimir force for Poschl-Teller potential in the Dirichlet limit. Exact vs. the harmonic oscillator approximation, eq. (4.8). On the vertical axis we plotted the ratio of the Casimir force and the harmonic oscillator approximation $a \to 0$ for $n = 1, 2, 3, 4$. For all the points we chose m^2 such that $V(0) + m^2 = 0$, so $x = 0$ is a turning point.

4.3 δ-function background

The case of a localized background potential reaches the extreme limit when $V(x) \to \beta\delta(x)$. The Casimir force here is a non-local effect, as can be understood from the fact that the potential vanishes (almost) everywhere, and the overall sign of the force cannot be predicted *a priori*. The WKB approximation as developed in the previous section includes only the local semiclassical modification of the Green's function, which vanishes in this case. There is a semi-classical method for this case (e.g. as $\beta, \lambda \to \infty$) which sums over classical paths that reflect from one surface to the other [17, 14] and accounts successfully for both the sign and the magnitude of the Casimir force. By applying the analysis of section 3.3 with $R(k) = \overline{R}(k) = \beta/(2ik - \beta)$, we obtain

$$\mathcal{F}(\lambda, \beta, m, a) = -\frac{\hbar c}{2\pi} \int_m^\infty \frac{\kappa d\kappa}{\sqrt{\kappa^2 - m^2}} \frac{\partial}{\partial a} \ln\left(1 + \frac{\lambda}{2\kappa}\left(1 - \frac{\beta}{2\kappa + \beta} e^{-2\kappa a}\right)\right). \tag{4.9}$$

The special case $\beta = \lambda$ has been studied in other contexts (see eq. (70) of ref. [18]), and our result agrees for that special case. For $\lambda \neq \beta$ however no new physics arises and we will not study that case here. More interesting is the limit $\lambda, \beta \to \infty$ with $m = 0$ in which case the integral can be performed yielding

$$\mathcal{F} = -\frac{\hbar c \pi}{24 a^2}, \tag{4.10}$$

the correct Casimir force for two impenetrable walls in one dimension at distance a from each other.

Another interesting limit is $\lambda \to \infty$, $\beta \to 0$ for $m = 0$ which makes contact with the Born approximation of section 3.4,

$$\mathcal{F}(\lambda, \beta, m, a)|_{m=0, \lambda\to\infty} = -\frac{\hbar c \beta}{4\pi a} + \mathcal{O}(\beta^2) \tag{4.11}$$

which agrees with the result obtained from eq. (3.25). Here we have a non-local force which is still a buoyancy force.

5. Beyond one dimension

Field theories become more divergent in higher dimensions. Casimir effects are no exception, indeed they are more problematic because both the loop divergences and the sharp background divergences become worse. Methods for renormalizing the loop divergences, at least in dimensions where the interaction with the background field is, in fact, renormalizable, have been worked out in refs. [5, 18, 16]. Divergences arising from the sharp background can be avoided by smoothing out the δ-function as described in ref. [5].

Once the buoyancy has been calculated in one dimension, the extension to higher dimensions can be constructed by the methods of ref. [6]. The core dynamics is the same in higher dimensions. The results can be summarized as follows: All the results for $n = 1$ generalize to any $n < 2$ without complication. For $2 \leq n < 3$ the generalization succeeds *except that the limit $\lambda \to \infty$ cannot be taken.* In that range of n the Casimir buoyancy depends on the cutoff, λ, on the strength of the boundary interaction. Thus there is no separation of scales, no "effective" low energy description of Casimir buoyancy. As $n \to 3$ two further complications arise: first, a new counterterm, $\mathcal{L}_{\rm CT} = c_2\sigma^2$, must be introduced to renormalize the two point (in the background field, σ) function. This means that the buoyancy depends on a renormalized coupling, actually the mass of the σ, that has to be specified. Second, as suggested by the appearance of an interaction proportional to $\sigma^2(x)$ induced by renormalization, the δ-function gives a divergent buoyancy. Instead it is necessary to smooth it out, replacing it for example, by a gaussian, as described in detail in ref. [5]. The buoyancy force then depends explicitly on the structure of the surface as well as the shape of the background. In the case of a real material one should make oneself sure of giving a proper description of the reaction of the material, using for example a plasma model for the metal with plasma frequency $\omega_{\rm plasma}$. In that case we expect that, if the $\lambda \to \infty$ and the $\omega_{\rm plasma} \to \infty$ limits exist then they coincide.

The problem of interest is an $n-1$ dimensional hyperplane immersed in a medium which is modulated in the $x_\perp$-direction. As shown in ref. [6], the Casimir energy (per unit 'area' of the $n-1$ transverse directions) for such an interface can be written as,

$$\mathcal{E}^n = \frac{1}{2}\int \frac{d^{n-1}p}{(2\pi)^{n-1}} \int_0^\infty dE \left(\sqrt{p^2+E} - \sqrt{p^2+m^2}\right)\left(\frac{dN}{dE} - \sum_{k=1}^{M} \frac{dN^{(M)}}{dE}\right) + \mathcal{E}_{\rm FD}(M) \tag{5.1}$$

where $\sqrt{E} = \sqrt{k^2+m^2}$ is the contribution to the energy from momentum in the $x_\perp$ direction. dN/dE is the density of states for the one dimensional problem and $dN^{(M)}/dE$ is the contribution to the density of states to $M^{\rm th}$ order in the background field, $\sigma(x) = \lambda\Delta(x-a) + V(x)$.[12] $\mathcal{E}_{\rm FD}(M)$ is the contribution of the Feynman diagrams through $M^{\rm th}$ order in the background field plus the contributions of counterterms necessary for renormalization.

[12] The "Levinson" subtraction discussed at length in ref. [6], is unnecessary here since the first Born subtraction does not contribute. To carry through the analysis for $n = 2$ a Levinson subtraction would be necessary.

The number of Born subtractions is determined by the degree of divergence of the field theory in σ with a $\sigma\phi^2$ coupling. For $n < 3$ only the tadpole diagram diverges, so only the first Born approximation must be subtracted. For the moment we set $N = 1$ and discuss the extension to $n = 3$ at the end of this section.

The p-integration in eq. (5.1) is to be understood in the sense of dimensional regularization, and can be performed,[13]

$$\mathcal{E}^n = -\frac{1}{2}\frac{\Gamma(-n/2)}{(4\pi)^{n/2}}\int_0^\infty dE\ E^{n/2}\left(\frac{dN}{dE} - \frac{dN}{dE}^{(1)}\right) + \mathcal{E}_{\rm FD}(1)\,. \tag{5.2}$$

The first Born approximation and the contribution of the tadpole diagram plus counterterm are linear in λ and linear in V. Therefore the V dependent term they generate is independent of a, so they do not contribute to the buoyancy and can be dropped,

$$\mathcal{F}^n = \frac{1}{2}\frac{\Gamma(-n/2)}{(4\pi)^{n/2}}\int_0^\infty dE\ E^{n/2}\frac{\partial^2 N}{\partial a \partial E}\,. \tag{5.3}$$

This remarkably simple form is valid for $n < 3$, and, of course, agrees with section 2 for $n = 1$.

As in one dimension, we take $\sigma(x) = \lambda\delta(x - a) + V$, construct the Greens function, rotate the integration to the negative E axis and integrate by parts,

$$\mathcal{F}^n = -\frac{1}{2\pi}\frac{\Gamma(1-\frac{n}{2})\sin\frac{n\pi}{2}}{(4\pi)^{n/2}}\int_0^\infty dE E^{\frac{n}{2}-1}\frac{\partial}{\partial a}\ln\left(1 + \lambda G_0(a, a, -E)\right). \tag{5.4}$$

This should be compared with eq. (2.15), which is the specialization to $n = 1$.

Note that the poles in $\Gamma(1 - n/2)$ are cancelled by the factor of $\sin n\pi/2$, which arose from the imaginary part of $(-E)^{n/2}$. Thus the only singularities in eq. (5.4) arise from divergences at the upper limit of the E integration. These get more serious as n increases because of the prefactor of $E^{n/2-1}$. It is easy to see that the E-integral in eq. (5.4) converges for $n < 3$: The high (negative) energy behavior of $G(a, a, -E)$ is dictated by the WKB approximation, eq. (3.4),

$$G(a, a, -E) \to \frac{1}{2\sqrt{E + W(a)}} + \mathcal{O}\left(E^{-1}\right) \tag{5.5}$$

(here again $W = m^2 + V$) and this behavior leads to a convergent integral for $n < 3$. However, if we attempt to take $\lambda \to \infty$ before doing the integral, we lose convergence already at $n = 2$.

Beyond eq. (5.4) the buoyancy depends on the specific form of $V(x)$. To get a feel for its variation, we evaluate the WKB approximation,

$$\mathcal{F}^n_{\rm WKB} = \frac{\lambda}{2(4\pi)^{\frac{n}{2}+1}}W^{\frac{n-3}{2}}\frac{dV}{da}\Gamma\left(1 - \frac{n}{2}\right)\sin\left(n\frac{\pi}{2}\right) f\left(\frac{\lambda}{2\sqrt{W(a)}}\right), \tag{5.6}$$

[13] Formally, the integral is defined and performed for values of $\mathrm{Re}\, n$ small enough that it converges. It is then analytically continued to real, positive n, with careful treatment of singularities encountered along the way.

where

$$f(x) = \int_0^\infty dy y^{n/2-1} \frac{1}{y+1} \frac{1}{\sqrt{y+1}+x} \,. \tag{5.7}$$

This function can be expressed by means of hypergeometric functions as

$$f(x) = -\frac{\pi}{x}\left(\left(1-x^2\right)^{\frac{n}{2}-1}-1\right)\csc\left(\frac{n\pi}{2}\right)+\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)\,F\left(1;\frac{3}{2}-\frac{n}{2},\frac{3}{2};x^2\right) \tag{5.8}$$

however does not carry much more information than the original integral, in particular because the interesting limit $\lambda \to \infty$ translates in $x \to \infty$ where the hypergeometric function has both a real and an imaginary part. The imaginary part is cancelled by the first term and the real part is the one we are interested in but this is difficult to isolate. However, before carrying on an asymptotic analysis of this integral it is straightforward to evaluate the $n = 2$ result

$$\mathcal{F}^{n=2}_{\rm WKB} = \frac{1}{16\pi}\frac{dV}{da}\log\left(1+\frac{\lambda}{2\sqrt{W(a)}}\right) = -\frac{d}{da}\Omega^{n=2}_{\rm WKB}\,, \tag{5.9}$$

where the quantum potential is

$$\Omega^{n=2}_{\rm WKB} = -\frac{1}{16\pi}\left(\frac{\sqrt{W}\,\lambda}{2}+\left(W-\frac{\lambda^2}{4}\right)\log\left(1+\frac{2\sqrt{W}}{\lambda}\right)-W\,\log\left(\frac{2\sqrt{W}}{\lambda}\right)\right). \tag{5.10}$$

Neither the quantum potential nor the force have a finite limit when $\lambda \to \infty$ in 2 dimensions. They diverge logarithmically in λ. The case $n = 1$ result coincides with the result of section 2.

Let us now return to the original integral representation to get an asymptotic expansion for large x (i.e. large λ). Writing it as

$$f(x) = \int_0^\infty dz e^{-zx} 2\int_1^\infty dy e^{-yz}\frac{1}{y}(y^2-1)^{n/2-1}\,, \tag{5.11}$$

performing the integral over y, expanding in series for small z and integrating term by term in z one gets[14]

$$\begin{aligned} f(x) \simeq\ & 2\pi\frac{1}{x^{3-n}}\csc(n\pi)+\frac{1}{x}\pi\csc\left(\frac{n\pi}{2}\right)+(2-n)\pi\frac{1}{x^{5-n}}\csc(n\pi)- \\ & -\frac{\Gamma(\frac{1}{2}-\frac{n}{2})\Gamma(\frac{n}{2})}{\sqrt{\pi}x^2}+\mathcal{O}\left(x^{-3}\right)+\mathcal{O}\left(x^{n-7}\right)\,. \end{aligned} \tag{5.12}$$

This expansion agrees very well with the exact expression eq. (5.8) already at $x \simeq 2$, as can be seen in figure 8, except in the limit $n \to 1$. The expansion (5.12) has a pole in $n = 1$ that goes like x^{-4}. This is not present in the original function f and indeed adding more terms pushes the pole to higher and higher powers of x^{-1}. However in this case we know the exact result either from section 2 (equation (3.5)) or by taking the limit $n \to 1$ of eq. (5.8) *before* expanding for $\lambda \to \infty$ (the two limits do not commute).

[14] Alternatively one could use the Mellin representation of the hypergeometric function [11] and move the contour of integration in order to pick the desired poles. This procedures is much more involved than the one described here and yields the same results.

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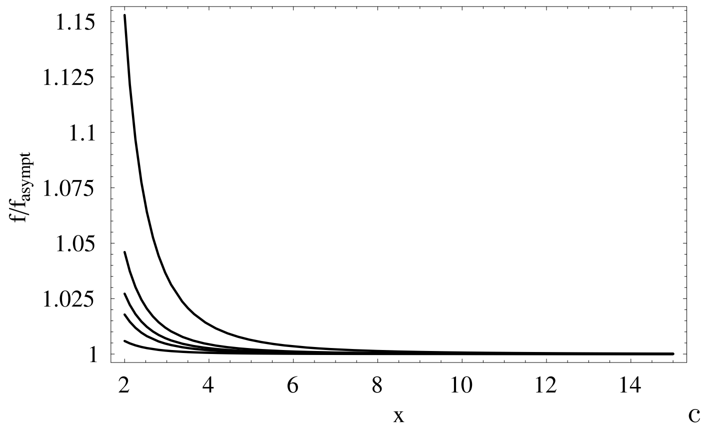

Figure 8: Comparison of f in eq. (5.8) with its asymptotic expansion eq. (5.12) for various n from up to down $n = 1.4, 1.8, 2, 2.2, 2.6$.

One thing to notice is that in passing from $n < 2$ to $n > 2$ the first term in eq. (5.12) becomes dominant over the second one. At exactly $n = 2$ they almost cancel each other leaving behind a term $\log(x)/x$ which reproduces the leading term in eq. eq. (5.9) (also the higher order terms agree with the expansion of eq. (5.9) for $\lambda \to \infty$). Another thing is that from this asymptotic expansion one can see that in the Dirichlet limit one has a finite force if and only if $n < 2$. For $n < 2$ the second term in eq. (5.12) is dominant over the first one and hence the force is independent of λ

$$\mathcal{F}_{\infty}^{n<2} \simeq \hbar c \frac{dV}{da} \frac{1}{4(4\pi)^{\frac{n}{2}}} W^{\frac{n-2}{2}} \Gamma\left(1 - \frac{n}{2}\right). \tag{5.13}$$

Having neglected the first term in eq. (5.12), however, this form cannot be used when $n \to 2$. Incidentally however notice that the limit $n \to 1$ now can be taken safely (and it gives the usual result eq. (3.6)) since we have not included the term $\propto \csc(n\pi)/x^{3-n}$ which generated pole in $n = 1$. This pole was a fiction of our procedure of taking the limit $\lambda \to \infty$ before $n \to 1$. It was signifying that for $n = 1$ the next term in the large x expansion of f decreases slower than $1/x^2$, indeed it is $\propto \log(x)/x^2$. As we said, including more and more terms in the asymptotic expansion pushes this pole further and further away.

As anticipated, for $n \geq 2$ the buoyancy force diverges when $\lambda \to \infty$. Keeping the first term in eq. (5.12) one obtains

$$\mathcal{F}_{\infty}^{n>2} \simeq \frac{\hbar c}{4^n \pi^{\frac{n}{2}}} \frac{dV}{da} \lambda^{n-2} \Gamma\left(1 - \frac{n}{2}\right) \sec\left(n\frac{\pi}{2}\right). \tag{5.14}$$

Notice that for $n \to 3$ we have a structure $\mathcal{F} \propto \frac{1}{n-3} \lambda dV/da$ which comes from a term $\propto \log\Lambda \int d^4x (V(x) + \lambda\delta(x-a))^2$ in the effective action (here Λ is the QFT cutoff). The pole $1/(n-3)$ from $\sec(n\pi/2)$ is the dimensional regularization way of seeing the logarithmic divergence $\log\Lambda$ in the two-legs graphs for $3+1$ dimensions.

The result eq. (5.6) (and its asymptotic expansions eq. (5.13) and eq. (5.14)) embodies all the behaviors we expect from the general analysis. It is a continuous function of n and finite for $n < 3$ and the Dirichlet limit, $\lambda \to \infty$, can only be taken for $n < 2$.

To go all the way to $n = 3$, the plane in three dimensional space, it would be necessary to invoke the full apparatus of the field theoretic approach to Casimir effects: the surface must be smoothed out and the second Born approximation to the density of states must be subtracted from the E-integration and added back together with the counterterm as a Feynman diagram.

6. Is Casimir buoyancy universal?

In the previous examples we have found Casimir buoyancy is ubiquitous. However all the previous examples share the fact that they can be very well described by the zero-order reflection term in the propagator. The WKB approximation was always an excellent approximation. There are obvious examples in which this is not true: Take for example the background V as made of a tall wall (a tall square wall will do the job) *plus* a shallow potential, decreasing toward the wall. If the shallow potential would not be present the delta function would be attracted toward the wall so for continuity this must be true for sufficiently small potentials *even if the total potential is decreasing toward the wall.* This is true because in the expression for the propagator $\mathcal{G}_0$ we cannot neglect the influence of the tall wall (that is why we called it 'tall') and this enters only through a non perturbative term in the form of the contribution from a closed path that goes from a to the wall and bounces back to a. More quantitatively by considering the background as made up of a wall and a smooth W, one can expand the propagator as [19]

$$\mathcal{G}_0(a, a, E) = \frac{i}{2\sqrt{E - W(a)}} + \frac{ie^{-i\pi\mu/2}}{2\sqrt{E - W(a)}} e^{iS_1(a,a,E)} \tag{6.1}$$

where $S_1(a, a, E)$ is the action of the closed path that goes from a to a bouncing on the wall and μ is the Maslov index of this orbit ($\mu = 1$ is the wall is 'smooth' on frequencies $\sqrt{E}$ and $\mu = 2$ for otherwise 'hard' walls). Inserting into eq. (2.14) one finds a general formula including both local (e.g. W and its derivatives) and non-local (depending on integrals of W or independent of W) contributions. The two effects interplay and they are difficult to separate but it is easy to extract the two limits: 1) When $W \to 0$ one recovers the Casimir attraction between impenetrable plates (compare with eq. (4.9) with $\beta \to \infty$) 2) When the wall is far away ($S_1 \gg 1$) the second term is exponentially smaller than the first and we find buoyancy from the first term only.

In general then a more correct statement would be that *Casimir buoyancy will occur in the cases where non-local effects, like closed orbits contributions to the propagator, are negligible.* We believe we have presented enough examples here to convince the reader that this is not a too stringent request.

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We believe that the response of a single boundary to an inhomogeneous background field was first considered for the case $V(x) \propto |x|$ in a paper begun in collaboration with one of the present authors (R.L.J.) ref. [20].

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